

# A new form of the no-cloning theorem for harmonic oscillator coherent states.

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We give an alternative formulation of the no-cloning theorem that applies to harmonic oscillator coherent states. It says that *unknown* single harmonic oscillator coherent states can not be *amplified*. Conversely it says that *known* harmonic oscillator coherent states can be directly amplified through an universal unitary process. It is also shown that such a formulation is needed for the consistency of *information cloning* proposed by us long ago.

## I. INTRODUCTION

The well known *no cloning theorem* [2] which asserts that in quantum theory it is *impossible* to make copies of an *unknown* single state is a remarkable aspect of quantum theory. It provides a remarkable consistency to the ensemble interpretation. For, if such a copying were possible, one could have created an *arbitrarily large* ensemble and determined the statistical significance through *ensemble measurements*. What is remarkable about this consistency is the fact that while one deals with unitary evolution, the other deals with quantum measurements!

In this brief note we give a somewhat more physically comprehensible formulation of the no-cloning theorem that is applicable to harmonic oscillator coherent states.

In this formulation the no-cloning theorem can be stated as 'It is impossible to *amplify* the excitation level of a single unknown harmonic oscillator coherent state. The single nature is crucial as if we had an ensemble of such states we could determine the state and apply suitable unitary transformations to obtain a state with scaled excitation. Of course there is no guarantee that such a unitary process would be *universal*.

In fact the following argument might indicate that such a universal process can not exist. Let us suppose we have two coherent states  $|\alpha\rangle, |\beta\rangle$  and let an universal unitary process  $U$  change their excitation (or equivalently the coherence parameters  $\alpha, \beta$  by a common (complex) scale  $\lambda$ . Then

$$\begin{aligned} U|\alpha\rangle &= |\lambda \cdot \alpha\rangle \\ U|\beta\rangle &= |\lambda \cdot \beta\rangle \end{aligned} \quad (1)$$

It then follows that

$$|\langle\alpha|\beta\rangle|^2 = |\langle\lambda \cdot \alpha|\lambda \cdot \beta\rangle|^2 \quad (2)$$

On noting the overlap for coherent states  $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$  it follows that  $\lambda = 1$ . This argument, though correct in itself, is not sufficient for our purposes. Not unexpectedly ancillary states are required to show that excitations can indeed be scaled. The *information cloning* protocol given by us long ago [3] is precisely what is required. That protocol can be stated as follows:

Consider a *single unknown* coherent state  $|\alpha\rangle$  and an ensemble of *known* coherent states which can also be displayed as a disentangled product state

$$|\Psi\rangle = |\alpha\rangle \cdot |\beta\rangle_1 \cdot |\beta\rangle_2 \cdots |\beta\rangle_N \quad (3)$$

The  $1 + N$  harmonic oscillators are described by the set of creation and annihilation operators  $(a, a^\dagger), (b_k, b_k^\dagger)$  (where the index  $k$  takes on values  $1, \dots, N$ ) satisfying the commutation relations

$$[a, a^\dagger] = 1; [b_j, b_k^\dagger] = \delta_{jk}; [a, b_k] = 0; [a^\dagger, b_k] = 0 \quad (4)$$

The (information)-cloning transformation is the universal unitary operator

$$U = e^{-\frac{\pi}{2\sqrt{N}}(a^\dagger \otimes \sum_j r_j b_j - a \otimes \sum_j r_j b_j^\dagger)} \quad (5)$$

The result of the operation of  $U$  on  $|\Psi\rangle$  is given by

$$U|\Psi\rangle = |-\sqrt{N}\beta\rangle \cdot \left|\frac{\alpha}{\sqrt{N}}\right\rangle_1 \cdots \left|\frac{\alpha}{\sqrt{N}}\right\rangle_N \quad (6)$$

This is what we have called *information cloning* earlier [3]. For coherent states complete information is contained in the parameter  $\alpha$ . The original unknown single state  $|\alpha\rangle$  has now been cloned into  $N$  identical copies of the state  $|\frac{\alpha}{\sqrt{N}}\rangle$ . By information cloning what we mean is this ability to make arbitrary number of copies of coherent states whose coherency parameter is  $c(N)\alpha$  where  $\alpha$  is the coherency parameter of the unknown coherent state and  $c(N)$  is a known constant depending on the number of copies made. It is not cloning because the  $N$ -copies are not identical to the original unknown state; yet the copies carry all the information about the original unknown state.

To elucidate matters further let us consider the action of  $U$  on another state  $|\Psi'\rangle$ :

$$|\Psi'\rangle = |\alpha'\rangle \cdot |\beta'\rangle_1 \cdot |\beta'\rangle_2 \cdots |\beta'\rangle_N \quad (7)$$

Then

$$U|\Psi'\rangle = |-\sqrt{N}\beta'\rangle \cdot \left|\frac{\alpha'}{\sqrt{N}}\right\rangle_1 \cdots \left|\frac{\alpha'}{\sqrt{N}}\right\rangle_N \quad (8)$$

We can then compare the squared inner products  $|\langle\Psi|\Psi'\rangle|^2$  and  $|\langle U\Psi|U\Psi'\rangle|^2$ :

$$\begin{aligned} |\langle\Psi|\Psi'\rangle|^2 &= |\langle\alpha|\alpha'\rangle|^2 \cdot |\langle\beta|\beta'\rangle|^{2N} \\ &= e^{-|\alpha-\alpha'|^2} \cdot e^{-N|\beta-\beta'|^2} \\ |\langle U\Psi|U\Psi'\rangle|^2 &= |\langle -N\beta | -N\beta'\rangle|^2 \cdot \left|\left\langle \frac{\alpha}{\sqrt{N}} \middle| \frac{\alpha'}{\sqrt{N}} \right\rangle\right|^{2N} \\ &= e^{-N|\beta-\beta'|^2} \cdot e^{-N \cdot \frac{1}{N}|\alpha-\alpha'|^2} \end{aligned} \quad (9)$$

Obviously the two expressions agree. It is worth examining closely the manner in which they agree with each other. The excitation of the unknown single state  $|\alpha\rangle$  has been split evenly among the  $N$ -states  $|\frac{\alpha}{\sqrt{N}}\rangle$ ; the excitations of the  $N$  known states  $|\beta\rangle$  have been merged into the excitation of the single known state  $|-N\beta\rangle$ . Thus the information cloning transformation of eqn(5) has *attenuated* the single unknown state while at the same time amplified the known state.

The latter fact gives rise to the converse statement of the new form of the no-cloning theorem for harmonic oscillator coherent states: 'It is possible to amplify known harmonic oscillator coherent states'. We see that in fact  $N$ -copies of such a known state are required. Since the state is known there is no problem in producing  $N$  identical copies of it.

If the new formulation of the no-cloning theorem was not correct, we would immediately run into a contradiction. Suppose, in opposition to the new form of the no-cloning theorem, it were possible to amplify an unknown coherent state through an universal unitary process (one is allowed to use further ancillaries), then the states  $|\frac{\alpha}{\sqrt{N}}\rangle$  could have been amplified to  $|\alpha\rangle$ . But then we would have succeeded in cloning the unknown single state  $|\alpha\rangle$ ! This would have violated the no-cloning theorem. Of course the transformation of eqn(5) did manage to amplify the state  $|\beta\rangle$ , but that was a *known* state.

We finally return to the consistency that the original no-cloning theorem gave to the ensemble interpretation of

quantum mechanics. Will the information cloning upset this consistency? This issue has been analyzed in [4]. It was found there that information cloning enables to determine the coherency parameter of the unknown coherent state via an ensemble measurement on the  $N$ -copies  $|\frac{\alpha}{\sqrt{N}}\rangle$  where the average value of the measurement correctly estimates  $\alpha$

$$\alpha_{est} = \alpha \quad (10)$$

but the measured variances are given by

$$\Delta\alpha_R = \Delta\alpha_I = \frac{1}{\sqrt{2}} \quad (11)$$

Thus, while the statistical error in usual measurements goes as  $\frac{1}{\sqrt{N}}$ , and can be made arbitrarily small by making  $N$  large enough, information cloning gives an error that is fixed and equal to the quantum mechanical uncertainty associated with the original unknown state.

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